

Set-valued Kurzweil-Henstock Integral in Riesz spaces

A. Boccuto–A. M. Minotti –A. R. Sambucini *

Abstract

A multivalued integral in Riesz spaces is given using the Kurzweil-Henstock integral construction. Some of its properties and a comparison with the Aumann approach are also investigated.

2010 AMS Mathematics Subject Classification: 28B15, 46G10.

Key words: Riesz space, Kurzweil-Henstock integral, Aumann integral, multivalued integral.

1 Introduction

Riesz spaces and set-valued integrals were studied by several authors and in different settings and have become an important tool in many branches of applied mathematics, in particular in Economic Theory (see [1]), where the theory of Riesz spaces is applied to study Riesz commodities and Riesz price economies, utility functions and equilibria in an exchange economy. Riesz spaces and ordered vector spaces provide a natural framework for any modern theory of integration: indeed in this context there are some relations with

*Supported by GNAMPA of CNR and University of Perugia

corresponding author: Anna Rita Sambucini

Authors' Address: Dipartimento di Matematica e Informatica, via Vanvitelli,1 I-06123 PERUGIA (Italy)

email: boccuto@dmf.unipg.it, minotti.am@gmail.com, matears1@unipg.it

Itô and Stratonovich-type integrals. Moreover Riesz spaces have been recently used in the setting of stochastic processes (see e.g. [17]): indeed they can be viewed as functions with values in the space $L^0(T, \Sigma, \mu)$ of all measurable functions with identification up to μ -null sets, where the convergence involved is the almost everywhere convergence, which is not generated by any topology. For a literature, see for example [3, 4, 5, 6, 15, 20, 22, 23, 24].

Multivalued analysis is a very powerful tool in the study of several problems in many areas of Mathematics. For example, we recall here subdifferentials of convex functionals, Calculus of Variations, degree theory, fixed points, set-valued random processes, optimal control theory, game theory, Pareto optimization, and so on.

The set-valued integration comes from Aumann using Bochner selections, and a very large number of generalizations of this approach were given. For example there are many recent studies concerning integrals of the type of Sugeno, Choquet, Pettis, Birkhoff, Kurzweil-Henstock, pseudo-integrals (see [9, 10, 12, 14]). Following the ideas in [16, 7, 8] we extend in two directions the results obtained. We introduce here a new kind of set-valued integral for Riesz space-valued multifunctions using the theory given in [5]: $\Phi(F, E)$ is in some sense the set of limits with respect to the (D) -convergence of all Riemann sums drawn out from γ -fine partitions. Its main properties like convexity, closedness, boundedness are investigated. We also prove that in the case of simple multifunctions this integral agrees with the usual integral obtained with Kurzweil-Henstock-type selections in the Aumann approach.

2 Preliminaries

Definition 1 Let R be a Riesz space. A bounded double sequence $(a_{i,j})_{i,j}$ in R is called a (D) -sequence or *regulator* if $a_{i,j} \geq a_{i,j+1}$ for all $i, j \in \mathbb{N}$ and $\bigwedge_{j=1}^{\infty} a_{i,j} = 0$ for all $i \in \mathbb{N}$.

A Riesz space R is said to be *Dedekind complete* iff every nonempty subset of R , bounded from above (below), has a lattice supremum (infimum) in R denoted by \bigvee (resp. \bigwedge). A Dedekind complete Riesz space R is said to be *super Dedekind complete* if every nonempty subset $R_1 \subset R$, $R_1 \neq \emptyset$, bounded from above, contains a countable subset having

the same supremum as R_1 .

A Dedekind complete Riesz space R is said to be *weakly σ -distributive* if for every (D) -sequence $(a_{i,j})_{i,j}$ in R one has:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \right) = 0. \quad (1)$$

We say that (R_1, R_2, R) is a *product triple* if there exists a map $\cdot : R_1 \times R_2 \rightarrow R$, which is compatible with the operations of sum and order, suprema and infima (see also [22, Assumption 5.2.1]).

A Dedekind complete Riesz space R is called an *algebra* if (R, R, R) is a product triple. Our results are given in (R, R, R) only for simplicity, but they hold in general in a product triple (R_1, R_2, R) .

We assume that (T, d) is a compact metric space, Σ is the σ -algebra of all Borel subsets of T and $\mu : \Sigma \rightarrow R$ is a positive additive mapping which is regular, namely: for every $E \in \Sigma$ there exists a (D) -sequence $(a_{i,j}^E)_{i,j}$ such that for every $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ there exist a compact set K and an open set U such that $K \subset E \subset U$ and

$$\mu(U \setminus K) \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}^E. \quad (2)$$

We shall see in a moment that under usual conditions a regular measure is also σ -additive.

Definition 2 A *gage* on T is a map $\gamma : T \rightarrow \mathbb{R}^+$.

Definition 3 A collection Π of T is a finite family $\Pi = \{(E_i, t_i) : i = 1, \dots, k\}$ of couples such that $t_i \in E_i$, $E_i \in \Sigma$ and $\mu(E_i \cap E_j) = 0$ for $i \neq j$. The points t_i , $i = 1, \dots, k$, are called *tags*. If moreover $\bigcup_{i=1}^k E_i = T$, Π is called a *partition*. Given a gage γ , we say that Π is γ -*fine* ($\Pi \prec \gamma$), if $d(w, t_i) < \gamma(t_i)$ for every $i = 1, \dots, k$ and $w \in E_i$.

A sequence $(r_n)_n$ in R (D) -converges to an element $r \in R$ if there exists a (D) -sequence $(a_{i,j})_{i,j}$ in R , such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists an integer n_0 such that $|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}$, for all $n \geq n_0$. In this case, we write $(D) \lim_n r_n = r$.

Observe that in weakly σ -distributive spaces the (D) -limits are unique.

We now recall the well-known Fremlin Lemma (see also [13]):

Lemma 4 *Let $(a_{i,j}^{(k)})_{i,j}$, $k \in \mathbb{N}$, be any countable family of (D) -sequences. Then for each fixed element $u \in R$, $u \geq 0$, there exists a (D) -sequence $(a_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, one has*

$$u \wedge \sum_{k=1}^s \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+k)}^{(k)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \quad \text{for all } s \in \mathbb{N}.$$

From now on, we shall always suppose that R is a weakly σ -distributive Dedekind complete algebra.

As announced before we can prove that

Theorem 5 *In a weakly σ -distributive Dedekind complete algebra every regular measure is σ -additive.*

Proof: Since μ is non-negative, all we must prove is that, for every disjoint sequence $(A_k)_k$ in Σ , one has

$$\mu\left(\bigcup_{k \in \mathbb{N}} A_k\right) \leq \sum_{k=1}^{+\infty} \mu(A_k).$$

To this aim, let us denote by A the union of all A_k 's, and observe that, for each integer k there exists a D -sequence $(a_{i,j}^k)_{i,j}$, such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ it is possible to find a compact set C_k and an open set U_k satisfying $C_k \subset A_k \subset U_k$ and $\mu(U_k \setminus C_k) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i+k)}^k$. Also, setting $u = 2\mu(\Omega)$, by lemma 4 it is possible to find a D -sequence $(b_{i,j})_{i,j}$ such that

$$u \wedge \sum_{k=1}^N \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+k)}^k \right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \quad \text{for all } N \in \mathbb{N}.$$

Now, by weak σ -distributivity and regularity, it is easy to deduce that $\mu(A) = \sup\{\mu(C) : C \subset A, C \text{ compact}\}$. Therefore, it will suffice to prove that

$$\mu(C) \leq \sum_{k=1}^{+\infty} \mu(A_k),$$

for every compact set $C \subset A$. So, fix any compact set $C \subset A$, and any mapping $\varphi \in \mathbb{N}^{\mathbb{N}}$. In correspondence with φ , there exist open sets U_k , $k = 1, 2, \dots$, such that $A_k \subset U_k$ and $\mu(U_k) \leq \mu(A_k) + \bigvee_{i=1}^{\infty} a_{i, \varphi(i+k)}^k$ for all k . Since C is compact, and the union of all U_k 's contains C , there exists an integer N such that $C \subset \bigcup_{k=1}^N U_k$. Since μ is non-negative we deduce:

$$\mu(C) \leq \sum_{k=1}^N \mu(U_k) \leq \sum_{k=1}^N \mu(A_k) + \sum_{k=1}^N \bigvee_{i=1}^{\infty} a_{i, \varphi(i+k)}^k = \mu\left(\bigcup_{k=1}^N A_k\right) + \sum_{k=1}^N \bigvee_{i=1}^{\infty} a_{i, \varphi(i+k)}^k,$$

and so

$$\mu(C) - \mu\left(\bigcup_{k=1}^N A_k\right) \leq u \wedge \sum_{k=1}^N \bigvee_{i=1}^{\infty} a_{i, \varphi(i+k)}^k \leq \bigvee_{i=1}^{\infty} b_{i, \varphi(i)}.$$

Then we have

$$\mu(C) \leq \mu\left(\bigcup_{k=1}^N A_k\right) + \bigvee_{i=1}^{\infty} b_{i, \varphi(i)} = \sum_{k=1}^N \mu(A_k) + \bigvee_{i=1}^{\infty} b_{i, \varphi(i)} \leq \sum_{k=1}^{+\infty} \mu(A_k) + \bigvee_{i=1}^{\infty} b_{i, \varphi(i)}.$$

But the last inequality holds for every $\varphi \in \mathbb{N}^{\mathbb{N}}$; so, since R is weakly σ -distributive, we finally get

$$\mu(C) \leq \sum_{k=1}^{+\infty} \mu(A_k),$$

as desired.

We now give our definition of Kurzweil-Henstock integrability.

Definition 6 [5, Definition 3.1] A function $f : T \rightarrow R$ is (KH) -integrable (or, in short, *integrable*) if there exist $I \in R$ and a (D) -sequence $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gage γ such that for every γ -fine partition $\Pi = \{(E_i, t_i), i = 1, \dots, q\}$ of T we have:

$$\left| \sum_{\Pi} f - I \right| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}, \quad (3)$$

where $\sum_{\Pi} f := \sum_{i=1}^q f(t_i) \mu(E_i)$ is a Riemann sum. The number I is determined uniquely. It will be denoted here simply by $\int_T f d\mu$. In an analogous way we can define the (KH) -integral on a set $E \in \Sigma$.

Observe that the (KH) -integral is additive on disjoint sets and is a linear positive functional (see [5, Propositions 3.2 and 3.5]). Moreover by [21, Lemma 1.10] the integrability on Σ -measurable subsets is inherited.

In Riesz spaces in general it is not possible to compare Bochner and Kurzweil-Henstock integrability, we only know that simple functions are integrable in both senses ([5, Theorem 3.7]); moreover it is possible to construct a Bochner integrable function which is not (KH) -integrable, as the following example shows.

Example 7 Let $R_2 = \mathbb{R}$ and $R = R_1 = c_{00}$ be the space of eventually null real-valued sequences. Let $(u_n)_n$ be defined by $u_n := (0, \dots, 0, 1, 0, \dots)$, where the value 1 is assumed at the n -th coordinate. The function $f : [0, 1] \rightarrow R$, defined by

$$f(x) = \begin{cases} u_n & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

vanishes almost everywhere (with respect to the Lebesgue measure), so has null Bochner integral, but is not (KH) -integrable on $[0, 1]$.

Indeed, fix arbitrarily $\delta : [0, 1] \rightarrow \mathbb{R}^+$ and $n \in \mathbb{N}$, $n \geq 2$. For every $i = 1, \dots, n-1$, let $\xi_i = \frac{1}{n+1-i}$ and choose an interval $]y_i, x_i[$ such that $\xi_i \in]y_i, x_i[$, $x_i - y_i < \delta(\xi_i)$, $[y_i, x_i] \cap [y_j, x_j] = \emptyset$ for all $i \neq j$, $0 < y_1$ and $x_{n-1} < 1$. We have:

$$0 < y_1 < x_1 < y_2 < x_2 < \dots < y_{n-1} < x_{n-1} < 1.$$

Let $x_0 = 0$, $y_n = 1$, and let us divide each of the intervals $[x_{i-1}, y_i]$, $i = 1, \dots, n$, in subintervals, in such a way to have a δ -fine partition: this is possible, by virtue of the Cousin Lemma ([18, Theorem 2.3.1]). These subintervals and the elements $([y_i, x_i], \xi_i)$, $i = 1, \dots, n-1$, form a δ -fine partition $\{([t_{j-1}, t_j], \eta_j) : j = 1, \dots, p\}$. Since $f = 0$ on each of the intervals $[x_{i-1}, y_i]$, $i = 1, \dots, n$, we have:

$$\sum_{j=1}^p (t_j - t_{j-1}) f(\eta_j) = \sum_{i=1}^{n-1} (x_i - y_i) f(\xi_i).$$

Let $\lambda_i^{(n)} = x_{n+1-i} - y_{n+1-i}$, $i = 2, \dots, n$: then we get

$$\sum_{j=1}^p (t_j - t_{j-1}) f(\eta_j) = \sum_{i=1}^{n-1} \lambda_{n+1-i}^{(n)} f(\xi_i) = \sum_{i=1}^{n-1} \lambda_{n+1-i}^{(n)} u_{n+1-i} \geq \lambda_n^{(n)} u_n.$$

Since $\lambda_n^{(n)}$ is strictly positive for every n , the sequence $\left(\sum_{\Pi(n)} f\right)_n$ is unbounded in R .

If f was (KH) -integrable on $[0, 1]$, then there would exist a gage $\delta_0 : [0, 1] \rightarrow \mathbb{R}^+$ such that

$$\bigvee \left\{ \sum_{\Pi} f : \Pi \text{ is a } \delta_0\text{-fine partition of } [0, 1] \right\} \in R :$$

this is a contradiction. Hence, f is not (KH) -integrable on $[0, 1]$. \square

3 Multivalued Kurzweil-Henstock integral

We introduce some definitions and properties.

Definition 8 Let $\mathcal{U}(C, r) := \{z \in R : \exists x \in C : |x - z| \leq r\}$. A set $C \subset R$ is said to be *closed* if $C = \text{cl}(C)$, where

$$\text{cl}(C) := \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \mathcal{U}(C, \bigvee_{i=1}^{\infty} a_{i,\varphi(i)})$$

Let $cf(R)$ be the family of all non-empty, convex, bounded and closed subsets of a weakly σ -distributive Riesz space R . For example every interval $[a, b]$ is closed. In fact, since $\bigwedge_{\varphi} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = 0$ for every (D) -sequence $(a_{i,j})_{i,j}$, then

$$\text{cl}(C) := \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} [a - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}, b + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}] = [a, b].$$

Analogously as in [11] the addition $+$: $cf(R) \times cf(R) \rightarrow cf(R)$ is defined as follows: it is the closure of the direct sum of the two sets. If A_i , $i = 1, \dots, n$ are in $cf(R)$ we denote by $\sum_{i=1}^n A_i$ the set

$$\sum_{i=1}^n A_i := \text{cl}(A_1 + \dots + A_n). \quad (5)$$

Observe that if $A, B \in cf(R)$ then $A + B \in cf(R)$. We have only to check the convexity. So, let $z_1, z_2 \in A + B$, we shall prove that for every real number $\alpha \in]0, 1[$,

$$\alpha z_1 + (1 - \alpha)z_2 \in A + B.$$

Let $(a_{i,j}^{(1)})_{i,j}, (a_{i,j}^{(2)})_{i,j}$ be two regulators associated with z_1, z_2 respectively, and set $b_{i,j} = 2(a_{i,j}^{(1)} + a_{i,j}^{(2)})$ for all $i, j \in \mathbb{N}$. It is easy to check that the double sequence $(b_{i,j})_{i,j}$ is a regulator. So for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $l = 1, 2$ there exist $x_l \in A, y_l \in B$ such that

$$|z_l - (x_l + y_l)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(l)}.$$

Then

$$|\alpha z_1 + (1 - \alpha)z_2| \leq |\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2)| + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

and this proves the convexity, since the direct sum is convex too.

Definition 9 A multifunction F is *bounded*, if there exists a positive element $L \in R$ with $F(t) \subset [-L, L]$ for all $t \in T$.

We now define a multivalued integral as follows:

Definition 10 Let $F : T \rightarrow 2^R$ be a multifunction, and $E \in \Sigma$. We call $(*)$ -integral of F on E the set

$$\begin{aligned} \Phi(F, E) = \{ & z \in R : \text{there exists } (a_{i,j})_{i,j} : \text{for all } \varphi \in \mathbb{N}^{\mathbb{N}} \text{ there is a gage } \gamma \\ & \text{such that for every } \gamma\text{-fine partition } P_\gamma := \{(E_i, t_i) : i = 1, \dots, k\} \in \Pi_\gamma \\ & \text{of } E \text{ there exists } c \in \sum_{i=1}^k F(t_i) \mu(E_i) \text{ with } |z - c| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \} \}. \end{aligned}$$

It is easy to check that, if F is single-valued and integrable, then its $(*)$ -integral coincides with the usual Kurzweil-Henstock integral given in [5]. Observe that, in the case of Example 7, we get $\Phi(\{f\}, E) = \emptyset$ since the Riemann sums are not bounded in R .

Then we have the following characterization:

Proposition 11 *The set $\Phi(F, E)$ can be obtained as follows:*

$$\Phi(F, E) = \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigcup_{\gamma} \bigcap_{\{(E_i, t_i)\} \in \Pi_{\gamma}} \mathcal{U} \left(\sum_{i=1}^k F(t_i) \mu(E_i), \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \right).$$

Moreover, if u is an upper bound for F , then $u \mu(T)$ is an upper bound for $\Phi(F, E)$ for every $E \in \Sigma$.

Proof: This is an easy consequence of definitions of $(*)$ -integral and of $\bigvee_{i=1}^{\infty} a_{i, \varphi(i)}$ -neighborhood. For the second part, thanks to boundedness and weak σ -distributivity, we have:

$$\begin{aligned} \Phi(F, E) &\subset \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigcup_{\gamma} \bigcap_{\{(E_i, t_i)\} \in \Pi_{\gamma}} \mathcal{U} \left([-u \mu(T), u \mu(T)], \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \right) \subset \\ &\subset \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} [-u \mu(T) - \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}, u \mu(T) + \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}] = [-u \mu(T), u \mu(T)]. \end{aligned}$$

□

We now prove convexity and closedness of the set $\Phi(F, E)$.

Proposition 12 *$\Phi(F, E)$ is convex provided that it is non-empty and the multifunction F is convex-valued.*

Proof: Let $z_1, z_2 \in \Phi(F, E)$; $(a_{i,j}^{(1)})_{i,j}, (a_{i,j}^{(2)})_{i,j}$ be two regulators associated with z_1, z_2 respectively, and set $b_{i,j} = 2(a_{i,j}^{(1)} + a_{i,j}^{(2)})$ for all $i, j \in \mathbb{N}$. Since z_1 and z_2 belong to $\Phi(F, E)$, then to every mapping $\varphi \in \mathbb{N}^{\mathbb{N}}$ there correspond two gages γ_1, γ_2 such that, whenever $\Pi := \{(G_l, \eta_l) : l = 1, \dots, q\}$ is both a γ_1 - and a γ_2 -fine partition, two points $c_1, c_2 \in \sum_{l=1}^q F(\eta_l) \mu(G_l)$ can be found, with

$$|z_s - c_s| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}^{(s)}, \quad s = 1, 2.$$

Let now $\gamma(x) := \gamma_1(x) \wedge \gamma_2(x)$, $x \in T$. Of course, every γ -fine partition is both γ_1 - and γ_2 -fine. Moreover, note that the set $\sum_{l=1}^q F(\eta_l) \mu(G_l)$ is convex, since F is convex-valued.

Hence, for every $\alpha \in [0, 1]$, $\alpha c_1 + (1 - \alpha) c_2 \in \sum_{l=1}^q F(\eta_l) \mu(G_l)$, and

$$\begin{aligned}
|\alpha c_1 + (1 - \alpha)c_2 - \alpha z_1 - (1 - \alpha)z_2| &\leq \alpha|c_1 - z_1| + (1 - \alpha)|c_2 - z_2| \leq \\
&\leq \alpha \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} + (1 - \alpha) \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} = \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.
\end{aligned}$$

Thus, $\alpha z_1 + (1 - \alpha) z_2 \in \Phi(F, E)$, which proves convexity of $\Phi(F, E)$. \square

Proposition 13 *If F is bounded and R is also super Dedekind complete, then the set $\Phi(F, E)$ is closed.*

Proof: We have to prove that

$$\Phi(F, E) = \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \mathcal{U}\left(\Phi(F, E), \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right).$$

One inclusion is obvious, for the converse let $z \in \bigcup_{(a_{i,j})_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \mathcal{U}\left(\Phi(F, E), \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right)$. Then there exists a regulator $(a_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $c_{\varphi} \in \Phi(F, E)$ with

$$|z - c_{\varphi}| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Since R is super Dedekind complete and weakly σ -distributive, a sequence $(\varphi_n)_n \in \mathbb{N}^{\mathbb{N}}$ can be found such that

$$0 = \bigwedge_{\varphi} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = \bigwedge_n \left(\bigvee_{i=1}^{\infty} a_{i,\varphi_n(i)} \right).$$

Without loss of generality, arguing as in [2], we can assume that $(\varphi_n)_n$ is increasing and $\varphi_n(i) \leq \varphi_n(i+1)$ for every $i, n \in \mathbb{N}$. Let now $z_n = c_{\varphi_n}$, so that $z = (o)\lim_n z_n = (D)\lim_n z_n$, where $z = (o)\lim_n z_n$ means that $\bigwedge_n \sup_{m \geq n} |z_m - z| = 0$.

We will show that $z \in \Phi(F, E)$. Observe that the sequence $(z_n)_n$ is bounded. Since $z_n \in \Phi(F, E)$, a regulator $(\alpha_{i,j}^{(n)})_{i,j}$ can be found, with the property that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$

there corresponds a gage γ_n such that for any γ_n -fine partition $\{(E_i, t_i) : i = 1, \dots, k\}$ there is $w_n \in \sum_{i=1}^k F(t_i) \mu(E_i)$ with

$$|z_n - w_n| \leq \bigvee_{i=1}^{\infty} \alpha_{i, \varphi(i+n)}^{(n)}$$

for all $n \in \mathbb{N}$. Since F is bounded, then $(w_n)_n$ is bounded, and hence the sequence $(z_n - w_n)_n$ is bounded too. Thus, by virtue of Lemma 4, there is a regulator $(\beta_{i,j})_{i,j}$ with

$$|z_n - w_n| \leq \bigvee_{i=1}^{\infty} \beta_{i, \varphi(i)}$$

for any $n \in \mathbb{N}$. Moreover, since the sequence $(z_n)_n$ (D) -converges to z , a regulator $(b_{i,j})_{i,j}$ can be found, with the property that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $\bar{n} \in \mathbb{N}$ such that

$$|z_n - z| \leq \bigvee_{i=1}^{\infty} b_{i, \varphi(i)} \quad \text{whenever } n \geq \bar{n}.$$

Let now $c_{i,j} = 2(\beta_{i,j} + b_{i,j})$, $i, j \in \mathbb{N}$. It is easy to see that $(c_{i,j})_{i,j}$ is a regulator.

The element $z_{\bar{n}}$ is such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gage $\gamma_{\bar{n}}$ with the property that for all $\gamma_{\bar{n}}$ -fine partition $\{(E_i, t_i) : i = 1, \dots, k\}$ there is $w_{\bar{n}} \in \sum_{i=1}^k F(t_i) \mu(E_i)$ with

$$|z_{\bar{n}} - w_{\bar{n}}| \leq \bigvee_{i=1}^{\infty} \beta_{i, \varphi(i)}$$

and

$$|z - w_{\bar{n}}| \leq |z - z_{\bar{n}}| + |z_{\bar{n}} - w_{\bar{n}}| \leq \bigvee_{i=1}^{\infty} c_{i, \varphi(i)}.$$

This implies that $z \in \Phi(F, E)$, and hence the set $\Phi(F, E)$ is closed. \square

Proposition 14 *If $0 \in F(t)$ for every $t \in T$, then*

$$\Phi(F, A) \subset \Phi(F, B) \text{ for every } A, B \in \Sigma, A \subset B.$$

Proof: Every element of the Riemann sum $\sum_{\Pi} F|_A$ can be extended to the Riemann sum $\sum_{\Pi} F|_B$, using [21, Lemma 1.2] (Cousin Lemma) and the fact that $0 \in F(t_i)$, for every

$t_i \in B \setminus A$. Now the proof follows using the definition of $(*)$ -integral. \square

The next goal, that we will obtain in the next section, is to prove that simple measurable multifunctions are integrable and the integrals can be obtained as in the single-valued case. We prove this fact in several steps. We begin with the following

Proposition 15 *Let C be a closed subset of R . If $F(t) \equiv C$ then, for every $E \in \Sigma$, we get*

$$\Phi(F, E) = \Phi(F1_E, T) = C\mu(E).$$

Proof: We will show that

$$C\mu(E) \subset \Phi(C1_E, T) \subset \Phi(C, E) \subset C\mu(E).$$

First of all observe that, since μ is regular, in correspondence with E there is a regulator $(a_{i,j}^E)_{i,j}$ such that for every $\varphi : \mathbb{N} \rightarrow \mathbb{N}$, there exist a compact set K and an open set U such that $K \subset E \subset U$ and satisfying (2). Arguing analogously as in [5], since K is compact and U is open, there exists a gage γ^E such that

$$B(t, \gamma^E(t)) := \{w \in T : d(t, w) < \gamma^E(t)\} \subset \begin{cases} U & \forall t \in K, \\ U \setminus K & \forall t \in U \setminus K, \\ T \setminus K & \forall t \notin U. \end{cases}$$

So, if $\Pi = \{(D_i, u_i), i = 1, \dots, q\}$ is a γ^E -fine partition of T , then

$$\mu(E\Delta(\cup\{D_i : u_i \in E\})) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^E.$$

Let $z \in C\mu(E)$ and consider $(a_{i,j}^E)_{i,j}$ and γ^E as above. Let $\Pi = \{(D_i, u_i), i = 1, \dots, q\}$ be a γ^E -fine partition of T , and $\tilde{D} = \cup_i \{D_i : u_i \in E\}$. Then

$$\sum_{i=1}^q C1_E(u_i)\mu(D_i) = \sum_{u_i \in E} C1_E(u_i)\mu(D_i) + \sum_{u_i \notin E} C1_E(u_i)\mu(D_i) = C\mu(\tilde{D}).$$

Since $\mu(\tilde{D}) \in [\mu(E) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^E, \mu(E) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^E]$, then there exists $c \in \sum_{i=1}^q C1_E(u_i)\mu(D_i)$ such that $|z - c| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^E$ and this proves the first inclusion.

If $z \in \Phi(F1_E, T)$, then there is a regulator $(b_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gage γ such that for every γ -fine partition $\Pi = \{(T_i, t_i) : i = 1, \dots, q\}$ there is $c \in \sum_{\Pi} F1_E$ such that $|z - c| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$. In particular this holds for γ -fine partitions $\Pi' = \{(D_i, t_i) : i = 1, \dots, s\}$ such that $\{(D_i, t_i) : t_i \in E\}$ are partitions of E . This proves that $z \in \Phi(F, E)$.

For the last inclusion, if we take $z \in \Phi(F, E)$, then there exists a regulator $(a_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gage γ such that for every γ -fine partition $\{(E_i, t_i) : i = 1, \dots, k\} \in \Pi_{\gamma}$ of E there is $c \in \sum_{i=1}^k F(t_i) \mu(E_i)$ with $|z - c| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$. Since the partition $\{(E_i, t_i) : i = 1, \dots, k\}$ is given for the set E , then the tags t_i are in E and so we get $F(t_i) = C$ for every $i = 1, 2, \dots, k$. Then $c \in C\mu(E)$ and $z \in \mathcal{U}(C\mu(E), \bigvee_{i=1}^{\infty} a_{i,\varphi(i)})$. Since this holds for every φ and C is closed, we get $z \in C\mu(E)$ and finally $\Phi(F, E) \subset C\mu(E)$ by arbitrariness of z , and this proves the last inclusion.

□

3.1 Comparison with the Aumann integral in Riesz spaces

We introduce now the Aumann integral via Kurzweil-Henstock integrable selections, in order to compare it with the previous integral for multifunctions. For a multifunction $F : T \rightarrow cf(R)$ let $S_F^1 = \{f : f(t) \in F(t) \mu - \text{a.e. and } f \text{ is } (KH)\text{-integrable}\}$ be the set of all (KH) -integrable selections of F .

Definition 16 If F is such that S_F^1 is non-empty, then for every $E \in \Sigma$ we define the *Aumann integral* (shortly *(A)-integral*) as

$$(A) \int_E F d\mu = \left\{ \int_E f d\mu, f \in S_F^1 \right\}.$$

We recall that, in the context of Banach spaces, the Aumann integral is defined via Bochner integrable selections. In our setting, this is not a good idea, as we showed in Example 7. As in the single-valued case we obtain that

Theorem 17 *If $F = \sum_{k=1}^n C_k 1_{E_k}$ is closed valued, $E_k \in \Sigma$ for all $k = 1, \dots, n$ and the E_k 's are pairwise disjoint, then for every $A \in \Sigma$ we get*

$$cl\left(\Phi\left(\sum_{k=1}^n C_k 1_{E_k}, A\right)\right) = \sum_{k=1}^n \Phi(C_k 1_{E_k}, A) = \sum_{k=1}^n C_k \mu(E_k \cap A) = cl\left((A) \int_A F d\mu\right).$$

Proof: First of all observe that in this case all the sets involved are non-empty. The equality $\sum_{k=1}^n \Phi(C_k 1_{E_k}, A) = \sum_{k=1}^n C_k \mu(E_k \cap A)$ follows immediately from Proposition 15 with $T = A$, $E = E_k$, $C = C_k$, $k = 1, \dots, n$. Moreover observe that, by (5), if $F = \sum_{k=1}^n C_k 1_{E_k}$, then for every gage γ and for all γ -partitions $\Pi = \{(B_r, t_r) : r = 1, \dots, w\}$ of A we have:

$$\begin{aligned} \sum_{\Pi} F &= \sum_{r=1}^w F(t_r) \mu(B_r) = \sum_{r=1}^w \left(\sum_{k=1}^n C_k 1_{E_k} \right) (t_r) \mu(B_r) = \\ &= \sum_{k=1}^n C_k \left(\sum_{r=1}^w 1_{E_k}(t_r) \mu(B_r) \right) = \sum_{k=1}^n \sum_{\Pi} F|_{E_k}. \end{aligned} \quad (6)$$

We now prove that

$$\Phi\left(\sum_{k=1}^n C_k 1_{E_k}, A\right) \subset \sum_{k=1}^n \Phi(C_k 1_{E_k}, A). \quad (7)$$

Let $z \in \Phi(\sum_{k=1}^n C_k 1_{E_k}, A)$, then there is a regulator $(b_{i,j})_{i,j}$ with the property that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gage γ such that for all γ -fine partitions $\Pi_* = \{(D_m, \xi_m) : m = 1, \dots, q\}$ of A there is $y \in \sum_{\Pi_*} (\sum_{k=1}^n C_k 1_{E_k})$ such that $|y - z| \leq \bigvee_{i=1}^{\infty} b_{i, \varphi(i)}$. We consider now only the γ -fine partitions $\Pi^* = \{(D'_m, t_m) : m = 1, \dots, s\}$ of A such that for every $k = 1, \dots, n$ the family $\Pi^*|_{E_k}$ is also a partition of E_k . By (6), we have that

$$y \in \sum_{m=1}^s \left(\sum_{k=1}^n C_k 1_{E_k} \right) (t_m) \mu(D'_m) = \sum_{k=1}^n C_k \left(\sum_{m=1}^s 1_{E_k}(t_m) \mu(D'_m) \right).$$

Thus for all $k = 1, \dots, n$ there are a regulator $(c_{i,j})_{i,j}$ and $y_k \in \Phi(C_k 1_{E_k}, A)$, for $k = 1, \dots, n$, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ we have $|y - \sum_{k=1}^n y_k| \leq \bigvee_{i=1}^{\infty} c_{i, \varphi(i)}$. This implies that

$$z \in \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \mathcal{U} \left(\sum_{k=1}^n \Phi(C_k 1_{E_k}, A), \bigvee_{i=1}^{\infty} 2(b_{i, \varphi(i)} + c_{i, \varphi(i)}) \right) \subset \bigcup_{a_{i,j}} \bigcap_{\varphi \in \mathbb{N}^{\mathbb{N}}} \mathcal{U} \left(\sum_{k=1}^n \Phi(C_k 1_{E_k}, A), \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \right).$$

Since $\sum_{k=1}^n \Phi(C_k 1_{E_k}, A)$ is closed, then the inclusion in (7) holds.

Obviously $\sum_{k=1}^n \Phi(C_k 1_{E_k}, A) = \sum_{k=1}^n C_k \mu(A \cap E_k) \subset (A) \int_A F d\mu \subseteq \text{cl} \left((A) \int_A F d\mu \right)$.

So we have only to prove that $(A) \int_A F d\mu \subset \Phi(F, A)$. But, if $z \in (A) \int_A F d\mu$, then $z = \int_A g d\mu$ with $g \in S_F^1$. By Definition 6, there exists a regulator $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gage γ such that for every γ -fine partition Π we have:

$$\left| \sum_{\Pi} g - z \right| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}. \quad (8)$$

Since $\sum_{\Pi} g \in \sum_{\Pi} F$, then the assertion follows. \square

Corollary 18 *Suppose that R is super Dedekind complete. If F is a simple multifunction with closed and bounded values then*

$$\Phi\left(\sum_{k=1}^n C_k 1_{E_k}, A\right) = \text{cl} \left((A) \int_A F d\mu \right).$$

Proof: it is an immediate consequence of Theorem 17 since $\Phi(\sum_{k=1}^n C_k 1_{E_k}, A)$ is closed thanks to Proposition 13. \square

It is still an open problem to compare the $(*)$ - and Aumann integrals for not necessarily simple multivalued functions.

References

- [1] Ch. D. ALIPRANTIS, O. BURKINSHAW, *Locally solid Riesz spaces with applications to Economics, Second edition*. Mathematical Surveys and Monographs, **105** (2003), Am. Math. Soc., Providence, RI.
- [2] A. BOCCUTO, D. CANDELORO, *Sobczyk-Hammer decompositions and convergence theorems for measures with values in l -groups*, Real Anal. Exch. **33** (2007/2008), 91-106.

- [3] A. BOCCUTO, D. CANDELORO, *Integral and ideals in Riesz spaces*, Inform. Sci. **179** (2009), 2891-2902.
- [4] A. BOCCUTO, D. CANDELORO, E. KUBIŃSKA, *Kondurar Theorem and Itô formula in Riesz spaces*, J. Concr. Appl. Math. **4** (2006), 67-90.
- [5] A. BOCCUTO, B. RIEČAN, *The Kurzweil-Henstock Integral for Riesz Space-Valued Maps Defined in Abstract Topological Spaces and Convergence Theorems*, PanAmerican Math. J. **16** (2006), 63-79.
- [6] A. BOCCUTO, B. RIEČAN, M. VRÁBELOVÁ, *Kurzweil-Henstock Integral in Riesz Spaces*, (2009), Bentham Science Publ.
- [7] A. BOCCUTO, A. R. SAMBUCINI, *A McShane Integral for Multifunctions*, J. of Concr. Appl. Math. **2** (4) (2004), 307-325.
- [8] A. BOCCUTO, A. R. SAMBUCINI, *A note on comparison between Birkhoff and McShane-type integrals for multifunctions*, Real Anal. Exch. (2011), to appear.
- [9] B. CASCALES, V. KADETS, J. RODRÍGUEZ, *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Anal. **256** (3) (2009), 673-699.
- [10] B. CASCALES, J. RODRÍGUEZ, *Birkhoff integral for multi-valued functions*, J. Math. Anal. Appl. **297** (2) (2004), 540-560.
- [11] C. CASTAING, M. VALADIER, *Convex analysis and Measurable multifunctions*, (1977), Lecture Notes in Math.
- [12] L. DI PIAZZA, K. MUSIAŁ, *Set-valued Kurzweil-Henstock-Pettis integral*, Set valued Anal. **13** (2005), 167-179.
- [13] D. H. FREMLIN, *A direct proof of the Matthes-Wright integral extension theorem*, J. London Math. Soc. **11** (1975), 276-284.

- [14] T. GRBIĆ, I. ŠTAJNER-PAPUGA, M. ŠTRBOJA, *An approach to pseudo-integration of set valued functions*, Inform. Sci. **181** (2011), 2278-2292.
- [15] D. HLINENÁ, M. KALINA, P. KRÁL', *Choquet integral with respect to Lukasiewicz filters, and its modifications*, Inform. Sci. **179** (2009), 2912-2922.
- [16] J. J. JARNIK, J. KURZWEIL, *Integral of multivalued mappings and its connection with differential relations*, Časopis pro pěstování matematiky **108** (1983), 8-28.
- [17] C. C. A. LABUSCHAGNE, B. A. WATSON, *Discrete stochastic integration in Riesz spaces*, Positivity **14** (4) (2010), 859-875.
- [18] L. P. LEE, R. VÝBORNÝ, *The integral: An easy approach after Kurzweil and Henstock*, (2000), Cambridge Univ. Press., Cambridge.
- [19] Y. OUYANG, R. MESIAR, J. LI, *An inequality related to Minkowski type for Sugeno integrals*, Inform. Sci. **180** (2010), 2793-2801.
- [20] E. PAP, M. ŠTRBOJA, *Generalization of the Jensen inequality for pseudo-integral*, Inform. Sci. **180** (2010), 543-548.
- [21] B. RIEČAN, *On the Kurzweil Integral in Compact Topological Spaces*, Radovi Mat. **2** (1986), 151-163.
- [22] B. RIEČAN, T. NEUBRUNN, *Integral, Measure and Ordering*, (1997), Kluwer Acad. Publ./Ister Science.
- [23] B. RIEČAN, M. VRÁBELOVÁ, *The Kurzweil construction of an integral in ordered spaces*, Czechoslovak Math. J. **48** (**123**) (1998), 565-574.
- [24] I. ŠTAJNER-PAPUGA, T. GRBIĆ, M. DANKOVÁ, *Pseudo Riemann-Stieltjes integral*, Inform. Sci. **180** (2010), 2923-2933.